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The dynamics of an elastic half-space under the action of a moving load☆

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Abstract

Fundamental solutions of a problem in the theory of elasticity are constructed for a half-space under the action of a load moving at constant velocity which does not change with time in a moving system of coordinates. On the basis of these solutions, the displacements of the medium are determined in the case of a load which moves along a cylindrical surface in the medium itself or over its boundary surface. Subsonic, transonic and supersonic cases are considered.

The investigation of the dynamics of extended underground structures under the action of perturbations leads to the solution of boundary value problems for continuous media with stress concentrators in the form of cylindrical cavities and inclusions. There is a fairly detailed bibliography on this question in Refs. 1,2. When the depth at which the structure is bedded is less than five times its characteristic diameters (for example, the tunnels of underground railways are often situated at such depths), it is necessary to take account of the closeness of the surface of the ground. Problems for an elastic half-space with a free boundary in the case of a load which moves at a specified constant velocity along a cylindrical surface within the medium and, also, problems of the dynamics of a multiply connected elastic half-space, which is weakened by cylindrical cavities of various shapes, under the action of loads which move along their surface are models for such investigations.

The dynamics of an elastic half-plane under the action of a load which moves at a constant velocity on its boundary have previously been investigated in Ref. 3. The fundamental solutions, in the three-dimensional case, of the equations of motion of an elastic medium under the action of a moving load concentrated on the axis were constructed in Ref. 4 and the method of boundary integral equations (MBIE) for solving boundary-value problems in the dynamics of an elastic medium with cylindrical cavities and boundaries under the action of surface loads which move at a constant velocity has been developed on the basis of these solutions.[5,*]

The construction of fundamental solutions for a half-space under the action of concentrated moving sources is not only useful from the point of view of the development of the MBIE for solving problems in the dynamics of an elastic half-plane with cylindrical cavities. Such solutions enable one to model the action of different distributed sources

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located both on the surface of the ground and close to it, to determine the stress and strain fields generated by them and to investigate wave processes. This last class of problems is typical in geophysics, seismology and the dynamics of underground constructions.

1. Formulation of the problem

An elastic isotropic medium D^- with Lamé parameters λ and μ and a density Δ occupies the half-space $x_1 > 0$. A load, which is constant in time and concentrated on the cylindrical surface $D \subset D^-$, $D = S \times (-\infty, \infty)$, the axis of which is parallel to the x_3 axis, moves at a constant velocity c along this axis and its components are represented in the form of the Fourier integral

$$P_i(x,z) = \sigma_{ij}n_j(x) = \int_{-\infty}^{\infty} p_i(x,\zeta)\exp(i\zeta z)d\zeta, \quad x \in S; \quad z = x_3 + ct$$
$$t \in (-\infty, \infty), \quad x = (x_1, x_2) \tag{1.1}$$

Suppose the components of the load are non-zero in a bounded set, that is, $\text{supp}_z P_i(x, z) \in (0, a)$, which corresponds to real physical problems. The boundary of the half-space is stress-free

$$\sigma_{j1} = 0 \text{ when } x_1 = 0, \quad j = 1, 2, 3 \tag{1.2}$$

where σ_{ij} are the components of the stress tensor, which are related to the components of the displacements u_k by Hooke's law

$$\sigma_{ij} = H_{ij}^k(\partial_1, \partial_2, \partial_3)u_k, \quad H_{ij}^k = \lambda\delta_{ij}\partial_k + (\delta_{ik}\partial_j + \delta_{jk}\partial_i) \tag{1.3}$$

δ_{ij} is the Kronecker delta, the subscripts i, j and k take the values 1, 2 and 3, $\partial_i = \partial/\partial x_i$, and summation is carried out from 1 to 3 in the case of repeated indices i, j and k.

We will assume that, in the case of fixed z, the components of the boundary load P_i are integrable on the contour S. If there are no bulk forces ($F = 0$), or they have a structure which is similar to the boundary load, the stresses and strains in the moving system of coordinates (x_1, x_2, z) satisfy the equations

$$\partial_j\sigma_{ij} - \rho c^2\partial_z^2 u_i + \rho F_i = 0, \quad i, j = 1, 2, 3; \quad \partial_z = \partial_3 \tag{1.4}$$

When account is taken of relations (1.3), we reduce Eq. (1.4) to the form

$$A_{ij}(\partial_1, \partial_2, \partial_z)u_j + c^{-2}F_i = 0$$
$$A_{ij}(\partial_1, \partial_2, \partial_z) = (M_1^{-2} - M_2^{-2})\partial_i\partial_j + (M_2^{-2}\partial_k\partial_k - \partial_z^2)\delta_{ij} \tag{1.5}$$

$M_j = c/c_j$ are the Mach numbers and $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the velocities of the bulk and shear waves in the elastic medium respectively.

We will consider three cases: subsonic ($c < c_2$), transonic ($c_2 < c < c_1$) and supersonic ($c > c_1$). In the first case ($M_1 < 1$, $M_2 < 1$), system (1.5) is of elliptic type, in the second case ($M_1 < 1$, $M_2 > 1$) the system is elliptic for the bulk component of the displacements and hyperbolic for the shear strains[2,4] and, in the supersonic case ($M_1 > 1$, $M_2 > 1$), the system is strictly hyperbolic. In the last two cases, shock waves can propagate if the following conditions are imposed on the discontinuities in the displacements and stresses in the fronts[5*]

$$[u_j] = 0, \quad [h_z\partial_j u_i - h_j\partial_z u_i] = 0, \quad [h_j\sigma_{ij} - \rho c^2 h_z\partial_z u_i] = 0 \tag{1.6}$$

where h is a wave vector. It is required to find the solution of the problem which satisfies the decay condition at infinity

$$u \to 0 \text{ when } R \to \infty, \quad R = \sqrt{x_1^2 + x_2^2 + z^2}, \quad x_1 \geq 0 \tag{1.7}$$

and certain radiation conditions which we shall introduce later in Section 3.

2. Green's tensor for an elastic half-space with a free boundary in the case of a moving load

In order to solve the problem, it is convenient to employ Green's tensor $V(x, z, y, \tau)$ for an elastic half-space with a free boundary, constructed in a moving system of coordinates. We have the following boundary-value problem to determine it: it is required to find, in the case of fixed $(y, \tau) = (y_1, y_2, \tau)$, $y_1 > 0$, the solution of the equations

$$A_{ij}(\partial_1, \partial_2, \partial_z)V_j^k = \delta_i^k\delta(x - y, z - \tau), \quad x_1 > 0 \tag{2.1}$$

which satisfies the conditions on the free surface

$$x_1 = 0: H_{i1}^k(\partial_1, \partial_2, \partial_z)V_k^m = 0, \quad m, i = 1, 2, 3 \tag{2.2}$$

and the radiation conditions at infinity.

We shall seek the solution in the form

$$V_j^i = U_j^i(x - y, z - \tau) + \Pi_j^i(x, y, z, \tau)$$

where U_j^i are the components of Green's tensor of Eq. (2.1) for an unbounded space corresponding to $F_j = \delta_j^i\delta(x_1, x_2, z)$ ($\delta(x_1, x_2, z)$ is a generalized singular δ-function). This tensor was constructed in Ref. 4, and its components have the form

$$U_j^i(x, z) = c_2^{-2}\delta_j^i f_{02}(r, z) + c^{-2}(\partial_i\partial_j f_{21}(r, z) - \partial_i\partial_j f_{22}(r, z)); \quad r = \|x\| = \sqrt{x_1^2 + x_2^2}$$

where

$$4\pi f_{0k}(r, z) = \frac{1}{V_k^+}, \quad 4\pi f_{2k}(r, z) = |z|\ln\frac{|z| + V_k^+}{m_k r} - V_k^+ + m_k r \quad \text{when} \quad c < c_k$$

$$2\pi f_{0k}(r, z) = \frac{\vartheta_k}{V_k^-}, \quad 2\pi f_{2k}(r, z) = \vartheta_k\left(z\ln\frac{z + V_k^-}{m_k r} - V_k^-\right) \quad \text{when} \quad c > c_k$$

$$2\pi f_{0k}(r, z) = -\delta(z)\ln r, \quad 2\pi f_{2k}(r, z) = z\theta(z)\ln r \quad \text{when} \quad c = c_k$$

$$V_k^{\pm} = \sqrt{z^2 \pm m_k^2 r^2}, \quad \vartheta_k = \theta(z - m_k r), \quad m_k = \sqrt{|1 - M_k^2|}$$

and $\theta(z)$ is Heaviside's function. Note that, at supersonic velocities, the carrier U is the interior of a cone with angle arctg $(1/m_1)$ at the vertex; supp $U = \{(x, z): z > m_1\|x\|\}$.

The tensor Π must satisfy the homogeneous equations of motion, the conditions for the radiation of waves at infinity and the following boundary conditions on the free surface

$$x_1 = 0: H_{i1}^k(\partial_1, \partial_2, \partial_z)\Pi_k^m = -\Sigma_{i1}^m(x - y, z - \tau), \ \Sigma_{i1}^m = H_{i1}^k(\partial_1, \partial_2, \partial_z)U_k^m(z - y, z - \tau) \tag{2.3}$$

It describes waves, reflected by the boundary of the half-space, which are generated by the action of a moving source concentrated at a point $x = y$, $z = \tau$. In order to construct this tensor, we will employ the vector-field potentials

$$\Pi_k^m = D_k^j(\partial_1, \partial_2, \partial_3)\Phi_j^m = \partial_k\Phi_1^m + e_{ki3}\partial_i\Phi_2^m + e_{kjl}e_{li3}\partial_i\partial_j\Phi_3^m, \quad k, m = 1, 2, 3 \tag{2.4}$$

Here e_{ijk} are the components of the unit skew-symmetric Levi-Civita pseudotensor. The first potential describes the gradient component of the field and the other two describe the vortex component. It can be shown that the potentials Φ_j^m satisfy the equations

$$\partial_k\partial_k\Phi_j^m - M_j^2\partial_z^2\Phi_j^m = 0, \quad j = 1, 2, 3 \tag{2.5}$$

By virtue of the linearity and homogeneity of the medium $\Pi_k^m(x, y, z, \tau) = \Pi_k^m(x, y, z - \tau)$.

In order to construct the solution, we will use the direct and inverse Fourier transforms of the tensors and potentials with respect to z (it is sufficient when $\tau = 0$):

$$\Pi_k^m = \int_{-\infty}^{\infty} \bar{\Pi}_k^m(x, y, \zeta)\exp(iz\zeta)d\zeta, \quad \bar{\Pi}_k^m = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi_k^m(x-y, z)\exp(-i\zeta z)dz$$

We define $\bar{U}_k^m$ and $\bar{\Phi}^m$ in a similar way.

In Fourier-transform space, the equations for the potentials take the form

$$(\partial_1^2 + \partial_2^2 \pm m_j^2\zeta^2)\bar{\Phi}_j^m = 0, \quad j = 1, 2, 3 \tag{2.6}$$

The plus sign corresponds to subsonic loads ($M_j < 1$) and the minus sign to supersonic loads ($M_j > 1$). The boundary conditions are transformed to the form

$$\begin{aligned} x_1 &= 0{:}\ B_j^k(\partial_1, \partial_2, i\zeta)\bar{\Phi}_k^m = -\bar{\Sigma}_{j1}(x-y, \zeta), \quad j = 1, 2, 3 \\ B_j^k &= H_{j1}^m(\partial_1, \partial_2, i\zeta)D_m^k(\partial_1, \partial_2, i\zeta) \end{aligned} \tag{2.7}$$

Hence, the problem of constructing the transform of the required tensors reduces to determining the potentials which satisfy Eq. (2.5), the boundary conditions on the free surface (2.7), certain radiation conditions and the decay conditions at infinity for the waves

$$\bar{\Phi}_j^k = o(r^{-(1+\varepsilon)}) \ \text{ when } \ r = \|x-y\| \to \infty \tag{2.8}$$

3. Determination of the potentials of the reflected waves

The potentials $\bar{\Phi}_j^k$ can be presented in the form of the superpositioning of the surface and plane waves which are reflected from the free surface of the half-space:

$$\bar{\Phi}_j^k = \int_{-\infty}^{\infty} \varphi_j^k(\eta, \zeta, y)\exp\left(ix_2\eta - x_1\sqrt{\eta^2 \pm m_j^2\zeta^2}\right)d\eta \tag{3.1}$$

$$\operatorname{Re}\sqrt{\eta^2 \pm m_j^2\zeta^2} \geq 0, \quad \operatorname{Im}\sqrt{\eta^2 \pm m_j^2\zeta^2} \leq 0 \tag{3.2}$$

The plus (minus) sign is taken if the load is subsonic (supersonic) for the corresponding value of j. It can be verified that potentials (3.1) satisfy Eq. (2.5). Conditions (3.2) are the radiation conditions for the waves reflected from the boundary of the half-space. The first of these conditions ensures the attenuation of the solutions at infinity and the second condition shows that the reflected waves move from the boundary of the half-space which corresponds to the physical representations.

The integrands in relation (3.1) can be found from boundary conditions (2.7), the right-hand sides of which also have to be expanded in Fourier integrals. When $x_1 = 0$, we have

$$\bar{\Sigma}_{j1}^k(x-y, \zeta) = \int_{-\infty}^{\infty} a_j^k(\eta, \zeta, y)\exp(ix_2\eta)d\eta \tag{3.3}$$

(concerning the definition of the integrands a_j^k, see below in Section 6).

Substituting expressions (3.1) and (3.3) into conditions (2.7), we obtain, for example, in the subsonic case

$$\int_{-\infty}^{\infty} [\mathrm{B}_k^j(-\sqrt{\eta^2 + m_j^2\zeta^2}, i\eta, i\zeta)\varphi_j^m + a_k]\exp(ix_2\eta)d\eta = 0, \quad k = 1, 2, 3$$

from which, by virtue of the arbitrariness of x_2, we conclude that the expression in the square brackets is equal to zero. As a result, we have a linear system of three equations for determining φ_j^m. Solving it, we find

$$\varphi_j^m = -a_k\Delta_{jk}^m(\eta,\zeta,y)/\Delta(\eta,\zeta), \quad \Delta(\eta,\zeta) = \det\left\{B_k^j\left(-\sqrt{\eta^2+m_j^2\zeta^2}, i\eta, i\zeta\right)\right\}$$

Here Δ is a Rayleigh determinant which, in this case, has the form

$$\Delta = 4\nu^2\sqrt{\nu^2-M_1^2\zeta^2}\sqrt{\nu^2-M_2^2\zeta^2}-(2\nu^2-M_2^2\zeta^2)^2, \quad \nu^2 = \zeta^2+\eta^2 \tag{3.4}$$

and Δ_{jk}^m is the corresponding cofactor. The properties of the Rayleigh determinant are well known. In particular, in the case being considered,

$$\Delta(\eta,\zeta) = 0 \quad \text{when} \quad \eta = \pm\zeta\sqrt{M_R^2-1} = \pm\eta_R, \quad M_R = c/c_R \tag{3.5}$$

where c_R, the velocity of the Rayleigh surface wave ($c_R < c_2$), is determined from the Rayleigh equation[6]

$$4\sqrt{1-\alpha_1^2}\sqrt{1-\alpha_2^2}-(2-\alpha_2^2)^2 = 0, \quad \alpha_j = c_R/c_j$$

When $c < c_R$, the determinant $\Delta(\eta,\zeta) \neq 0$ for any real η and ζ. Using formulae (3.1) and (2.4), we construct in this case the potentials and displacements of the medium V_i^j. All the integrands are continuous and tend quite rapidly to zero at infinity which can be shown using the property of the boundedness of the carrier of a moving load. Hence, the integrals exist and they satisfy the decay conditions at infinity. The solution has therefore been constructed in this case.

When $c_R < c < c_2$, a solution of Eq. (3.4) exists but there are no solutions which decay at infinity. If we drop the decay condition for the displacements behind the moving load (when $z > 0$) then, to construct the solution, it is necessary to transform the integration contour in the neighbourhood of the points $-\eta_R$ and $+\eta_R$ by passing around them in the upper and lower half-planes of the complex η-plane where radiation conditions (3.2) are satisfied. Using Jordan's lemma for integration in the complex plane, we obtain

$$\bar{\Phi}_j^k = \text{v.p.}\int_{-\infty}^{\infty}\varphi_j^k(\eta,\zeta,y)\exp\left(ix_2\eta - x_1\sqrt{\eta^2+m_j^2\zeta^2}\right)d\eta +$$

$$+\sum_{\pm}\left[\pm\pi i\,\text{Res}\,\varphi_j^k(\pm\eta_R,\zeta,y)\exp\left(\pm ix_2\eta_R - x_1\sqrt{\eta_R^2+m_j^2\zeta^2}\right)\right]$$

where Res$f$ is the residue of the function at the above-mentioned point. The two terms in the last summation correspond to the upper and lower sign and describe Rayleigh surface waves, which the moving load generates in this case. Note that the potentials do not decay behind the load at infinity:

$$\bar{\Phi}_j^k \sim 2\pi i\int_{-\infty}^{\infty}\text{Res}\,\varphi_j^k(\pm\eta_R,\zeta,y)\exp\left(-iz\zeta \pm ix_2\eta_R - x_1\sqrt{\eta_R^2+m_j^2\zeta^2}\right)d\zeta \quad \text{when} \quad z\to+\infty$$

The plus sign is taken when $x_2 > 0$ and the minus sign when $x_2 < 0$.

When $c = c_R$, the integrands in expression (3.1) also have strong non-integrable singularities in the sense of a principal value and, in this case, a steady-state solution of the problem does not exist.

The calculations are carried out in an analogous manner for the supersonic and transonic cases for which $\Delta(\eta, \zeta) \neq 0$ for all $\eta \in (-\infty, \infty)$.

4. Fundamental spatially-periodic solutions

In order to determine the Fourier transforms of the boundary functions when $x_1 = 0$, we will determine the Fourier transform of the tensor U with respect to z. To do this, we employ the complete Fourier transform of this tensor, which has previously been constructed in Ref. 4. Its incomplete transform with respect to two of the Fourier variables corresponding to x_1 and x_2 enables us to construct the Fourier transform of the tensor U with respect to z:for subsonic speeds

$$2\pi\bar{U}_i^j(x,\zeta) = c_2^{-2}\delta_i^j K_0(m_2|\zeta|r) + (c\zeta)^{-2}\partial_i\partial_j(K_0(m_2|\zeta|r) - K_0(m_1|\zeta|r))$$

and, for transonic and supersonic speeds,

$$4i\bar{U}_i^j(x,\zeta) = c_2^{-2}\delta_i^j H_0(m_2|\zeta|r) + (c\zeta)^{-2}\partial_i\partial_j(H_0(m_2|\zeta|r) - F(m_1|\zeta|r))$$

where $F(\xi) = 2i\pi^{-1}K_0(\xi)$ for transonic speeds and $F(\xi) = H_0(\xi)$ for supersonic speeds, K_0 is MacDonald's function, H_0 is the Hankel function of the second kind ($H_0^{(2)}$ when $\zeta > 0$) and of the first kind ($H_0^{(1)}$ when $\zeta < 0$) and $r = \sqrt{x_1^2 + x_2^2}$. Unlike the symbols for the derivatives ∂_1 and ∂_2, the symbol ∂_3 corresponds to multiplication by $i\zeta$.

In the subsonic case, the function $K_0(mr|\zeta|\exp(i(x_3 + ct))$ describes waves which decay exponentially as $r \to \infty$ and propagate along the z axis (cylindrical surface waves). In the supersonic case, the choice of the Hankel functions is associated with the sign of the index of the exponent $\exp(i\zeta ct)$ since it is well known[6] in this case that the above mentioned functions just describe the potentials of waves which satisfy the Sommerfeld radiation conditions at infinity.

5. Fundamental stress tensors

Using Hooke's law (1.3), we introduce the stress tensors which are generated by the tensor $U(x - y, z)$ with the components

$$S_{ij}^k(x,y,z) = H_{ij}^m(\partial_1,\partial_2,\partial_3)U_m^k(x-y,z), \quad \Gamma_i^k(x,y,z,n) = S_{ij}^k(x,y,z)n_j$$
$$T_i^k(x,y,z,n) = \Gamma_k^i(y,x,z,n) \tag{5.1}$$

where[4]

$$\frac{2\pi c^2}{\mu}T_j^i(x,y,z,n) = (2M_1^2 - M_2^2)n_j\partial_i f_{01} -$$
$$- M_2^2(\delta_j^i n_k\partial_k f_{02} + n_i\partial_j f_{02}) - 2n_k\partial_i\partial_j\partial_k(f_{01} - f_{02}) \tag{5.2}$$

$$T_i^j(x,y,z,n) = -T_i^j(y,x,z,n) = -T_i^j(x,y,z,-n) \quad \text{for any} \quad c \tag{5.3}$$

and the following lemma is true (see the preprint mentioned in Footnote 1).

Lemma 1. *For a fixed k, the tensor T with the components T_i^k is the basic solution of* Eq. (1.5) *in the case when the components of the body force have the form*

$$F_i^k = \lambda n_i\partial_k\delta + \mu(\delta_i^k n_j\partial_j\delta + n_k\partial_i\delta) \tag{5.4}$$

where $\delta = \delta(x_1 - y_1, x_2 - y_2, z)$.

Convolution of the tensor U with the characteristic function of the set $H_G^-(x, z)$, where G^- is any domain bounded by a Lyapunov surface G, enables us to obtain a formula which is similar to the Gauss formula for the potential of the double layer of the Laplace equation.

Lemma 2. *When $c < c_2$, the tensor T satisfies the relation*

$$\rho\delta_i^j H_G^-(x,z) = \int_G T_i^j(y-x,\tau-z,n(y,\tau))dS(y,\tau) +$$
$$+ \rho c^2\int_G U_{i,z}^j(y-x,\tau-z)n_z(y,\tau)dS(y,\tau)$$

For $(x, z) \in G^-$, both integrals are regular and $H_G^- = 1$. For $(x, z) \in G$, the first integral is singular. It is taken in the sense of a principal value and $H_G^- = 1/2$.

If $G = D$ is a cylindrical surface, the second integral is equal to zero and formula (5.4) is analogous to the Gauss formula in the static theory of elasticity.[7]

The tensor T plays a fundamental role in the construction of the boundary integral equations (BIE) for solving boundary-value problems in the case of moving loads in cylindrical cavities in elastic media. In the case of transonic and supersonic velocities, see Ref. 5 concerning the construction of the BIE of boundary-value problems.

6. Stresses in the boundary of a half-space

In order to construct the tensor V, it is necessary to determine the integrands in formula (3.3). Taking account of relations (5.1), (5.2) and (2.3), we define the left-hand side of equality (3.3) by the following expression when $x_1 = 0$

$$\frac{2\pi c^2}{\mu}\bar{\Gamma}_i^j = (2M_1^2 - M_2^2)\delta_1^j\partial_i\bar{f}_{01} - M_2^2(\delta_i^j\partial_1\bar{f}_{02} + \delta_1^i\partial_j\bar{f}_{02}) + 2\partial_1\partial_i\partial_j(\bar{f}_{01} - \bar{f}_{02})$$
$$\bar{f}_{0j} = K_0(m_j|\zeta|r)/(2\pi), \quad c < c_j; \quad \bar{f}_{0j} = iH_0(m_j|\zeta|r)/4, \quad c > c_j \tag{6.1}$$

To represent expression (6.1) in the form of the Fourier integrals (3.3), we next use a plane-wave expansion of cylindrical functions,[1] that is, when $x_1 - y_1 < 0$, we have

$$K_0(kr) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\chi(\eta, k^2)}{\sqrt{\eta^2 + k^2}}d\eta, \quad H_0^{(1)}(kr) = \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{\chi(\eta, -k^2)}{\sqrt{\eta^2 - k^2}}d\eta \tag{6.2}$$

Here $\chi(\eta, k^2) = \exp(ix_2\eta + (x_1 - y_1)\sqrt{\eta^2 + k^2})$, $H_0^{(2)}(kr) = \overline{H_0^{(1)}}(kr)$, where a bar over a function is the sign of the complex conjugate. This enables us to represent the boundary stresses in the form of the Fourier integrals (3.3). To do this, we substitute expansion (6.2) corresponding to a given velocity into expression (6.1) using the representations

$$\partial_1\bar{f}_{0j} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\chi(\eta, \pm\zeta^2 m_j^2)d\eta, \quad \partial_2\bar{f}_{0j} = \frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{\eta\chi(\eta, \pm\zeta^2 m_j^2)}{\sqrt{\eta^2 \pm \zeta^2 m_j^2}}d\eta \tag{6.3}$$

We obtain the second derivative by differentiating the integrals (6.3). We then group the terms accompanying $\exp(ix_2\eta)$ which are also the integrands a_j^k in formula (3.3) if we put $x_1 = 0$.

The problem of constructing Green's tensor for an elastic half-space is therefore solved.

7. Moving surface loads

The action of moving loads on a cylindrical surface D in an elastic block can be replaced by a body force F which is described by a singular generalized function, that is, by a simple layer on the surface D with components $F_i = p_i(x, z)\delta_D(x, z)$. Then, using the properties of Green's tensor, we obtain the solution of the problem

$$u_i = \int_S dS(y)\int_{-\infty}^{\infty} V_i^k(x, y, z - \tau)p_k(y, \tau)d\tau \tag{7.1}$$

If D is the surface of a cylindrical cavity in an elastic half-space, formula (7.1) provides a good description of the displacement of the block at a sufficient distance from the cavity. In particular, it can be used to estimate the stress-strain state of the surface of the ground ($x_1 = 0$) along deeply laid underground train tunnels (at a depth greater than five times the characteristic diameters of the cavity).

8. Moving loads on the surface of the ground

In the case of moving loads on the surface of the half-space $x_1 = 0$ it is necessary to construct the solution of the homogeneous Eq. (1.5) under the action of a concentrated boundary force with components of the form $P_j(x_2, z) = \delta_j^m \delta(x_2, z)$. The procedure for constructing the solution is exactly the same as the procedure for constructing the tensor Π if we take

$$x_1 = 0: H_{j1}^k(\partial_1, \partial_2, \partial_3)\Pi_k^m = \delta_j^m \delta(x_2, z) \tag{8.1}$$

instead of boundary condition (2.3). In this case, the integrands, which are analogous to those in formula (3.3), are easily determined: $a_j^k = (2\pi)^2 \delta_j^k$. It is necessary to use these when determining of the potentials in formulae (3.3).

The solution of the problem in the case of integrable loads with a finite carrier on the surface of the ground can be represented in the form of a Duhamel integral

$$u_j = \int_S dS(y) \int_{-\infty}^{\infty} \Pi_j^k(x, y, z-\tau) p_k(y, \tau) d\tau$$

9. Conclusion

The method of boundary integral equations (BIE) for an elastic space with a cylindrical cavity under the action of moving loads, developed in Ref. 5, can be used to estimate the stress-strain state of a large mass in the neighbourhood of a shallowly laid tunnel. Here, the need arises to solve the singular BIE for the union of two surfaces: the boundary of the cavity and the plane boundary of the half-space. If the fundamental solutions for the half-space constructed here are used as kernels to construct the BIE, then the BIE are only solved for the surface of the cavity. At the same time, the number of boundary elements and, correspondingly, the order of the linear systems of equations used in the discrete analogues of the BIE to solve them, is significantly reduced. However, in the computational scheme, this is a more labourious procedure since, in order to determine Green's tensor, it is necessary to calculate Fourier integrals at each point of integration while the fundamental solutions for an elastic space are calculated using simple analytical formulae.

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